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CHARACTERISTIC MATRICES OF SOME HYBRID CELLULAR AUTOMATA WITH RULES 60 AND 102

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Abstract. We investigate periodicities of characteristic matrices of some hybrid cellular automata configured with rules 60 and 102 and an intermediate boundary condition.

1. Introduction

Properties of cellular automata with an intermediate boundary condition have been studied some researchers [1,2,5-7]. Recently, some periodicities of characteristic matrices of cellular automata configured with rule 60 and an intermediate boundary condition was investigated [3,4].

In this note, we will investigate periodicities of characteristic matrices of some hybrid cellular automata configured with rules 60 and 102 and an intermediate boundary condition.

2. Preliminaries

In this section, we will introduce some terminologies shall be used in this note.

A cellular automaton (CA) is an array of sites (cells) where each site is in any one of the permissible states. At each discrete time step (clock cycle) the evolution of a site value depends on some rules (the combinational logic) which is a function of the present states of its k neighbors for a k-neighborhood CA. For a 2-state 3-neighborhood CA, the evolution of the (i)th cell can be represented as a function of the present states of $(i-1)$th, (i)th, and $(i+1)$th cells as: $x_i(t+1) = f\{x_{i-1}(t), x_i(t), x_{i+1}(t)\}$, where f represents the combinational logic. For such a CA, the modulo-2 logic is always applied.

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For a 2-state 3-neighborhood CA there are 2^3 distinct neighborhood configurations and 2^{2^3} distinct mappings from all these neighborhood configurations to the next states, each mapping representing a CA rule. The CA, characterized by a rule known as rule 60, specifies an evolution from the neighborhood configurations to the next states as;

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} .$$

The rule name 60 comes from that 00111100 in a binary system is 60 in a decimal system. The corresponding combinational logic of rule 60 is

$$x_i(t+1) = x_{i-1}(t) \oplus x_i(t),$$

that is, the next state of (i)th cell depends on the present states of its left and self neighbors.

And the CA, characterized by a rule known as rule 102, specifies an evolution from the neighborhood configurations to the next states as;

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array} .$$

The rule name 102 comes from that 01100110 in a binary system is 102 in a decimal system. The corresponding combinational logic of rule 102 is

$$x_i(t+1) = x_i(t) \oplus x_{i+1}(t),$$

that is, the next state of (i)th cell depends on the present states of self and its right neighbors.

If in a CA the same rule applies to all cells, then the CA is called a uniform CA; otherwise the CA is called a hybrid CA. There can be various boundary conditions; namely, null (where extreme cells are connected to logic '0'), intermediate (where the 2nd right cell of the leftmost cell of a 3-neighborhood CA is assumed to be the left neighbor of the leftmost cell of the CA and the 2nd left cell of the rightmost cell of the CA is assumed to be the right neighbor of the rightmost cell of the CA), periodic (where extreme cells are adjacent), etc. The number of cells of a CA is called the length of a CA.

The characteristic matrix T of a CA is the transition matrix of the CA. The next state $f_{t+1}(x)$ of a linear CA is given by $f_{t+1}(x) = T \times f_t(x)$, where $f_t(x)$ is the current state and t is the time step.

3. Characteristic matrices of hybrid cellular automata

We begin with characteristic matrices of a hybrid CA configured with rules 60 and 102 and an intermediate boundary condition. Such a matrix T of the CA H of length $l(\geq 6)$ where the rule applied to the first n cells of H is 60 and the rule applied to the remaining cells is 102 with $3 \leq n \leq l-3$ is given by

$$T_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } 1 < i \leq n \text{ and } j = i-1, \\ 1, & \text{if } n < i < l \text{ and } j = i+1, \\ 1, & \text{if } i = 1 \text{ and } j = 3, \\ 1, & \text{if } i = l \text{ and } j = l-2, \\ 0, & \text{otherwise} \end{cases}$$

or

$$T = \begin{pmatrix} 1 & 0 & 1 & & & & & & & & & & & \\ 1 & 1 & 0 & & & & & & & & & & & \\ 0 & 1 & 1 & & & & & & & & & & & \\ & & 1 & 1 & & & & & & & & & & \\ & & & \cdot & \cdot & & & & & & & & & \\ & & & & \cdot & \cdot & & & & & & & & \\ & & & & & \cdot & \cdot & & & & & & & \\ & & & & & & 1 & 1 & 0 & & & & & \\ & & & & & & & 0 & 1 & 1 & & & & \\ & & & & & & & & & \cdot & \cdot & & & \\ & & & & & & & & & & \cdot & \cdot & & \\ & & & & & & & & & & & \cdot & \cdot & \\ & & & & & & & & & & & 1 & 1 & \\ & & & & & & & & & & & & 1 & 1 & 0 \\ & & & & & & & & & & & & 0 & 1 & 1 \\ & & & & & & & & & & & & 1 & 0 & 1 \end{pmatrix}$$

where the all values of the blank entries are zero. From now on, the all values of the blank entries in matrix representation will always be zero unless otherwise specified.

We can obviously know that the first n rows and the remaining rows of such a matrix T are completely independent each other in any iteration of multiplication of T. So we can have a corollary by recalling two results in [4].

THEOREM 3.1. [4] *Let T be the characteristic matrix of a uniform CA of length n with $3 \leq n \leq 3 + 2^t$ for some non-negative integer t configured with rule 60 and an intermediate boundary condition. Then $(T^{2^t \cdot 3})^2 = T^{2^t \cdot 3}$ and $T^{1+2^t \cdot 3} = T$.*

THEOREM 3.2. [4] *Let T be the characteristic matrix of a uniform CA of length n with $3 + 2^{t-1} < n \leq 3 + 2^t$ for some positive integer t configured with rule 60 and an intermediate boundary condition. Then $T^{1+m} = T$ for some positive integer m if and only if m is a multiple of $2^t \cdot 3$.*

COROLLARY 3.3. *Let T be the characteristic matrice of a hybrid CA of length $l (\geq 6)$ configured with rule 60 for the first n cells and rule 102 for the remaining $l - n$ cells and an intermediate boundary condition where $3 \leq n < l - 3$. And let s and t be positive integers so that $3 + 2^{s-1} < n \leq 3 + 2^s$ and $3 + 2^{t-1} < l - n \leq 3 + 2^t$. Then:*

(1) $(T^{2^r \cdot 3})^2 = T^{2^r \cdot 3}$ *where* $r = \max\{s, t\}$.
(2) $T^{1+m} = T$ *for some positive integer m if and only if m is a multiple of $2^r \cdot 3$ where $r = \max\{s, t\}$.*

Proof. The first n rows with rule 60 and the remaining $l-n$ rows with rule 102 of T are completely independent each other in any iteration of multiplication of T as mentioned above. Since rules 60 and 102 are symmetric each other, the first n rows and the remaining $l - n$ rows are symmetric each other if their sizes are ignored. So Theorem 3.1 and 3.2 can be applied not only to the first n rows but also to the remaining $l - n$ rows. Thus we have the results. □

Next, we deal with the characteristic matrix T of a hybrid CA H of length $l (\geq 4)$ with an intermediate boundary condition where the rule applied to the first $l - 1$ cells of H is 60 and the rule applied to the last 1 cell of H is 102. Such a matrix T is given by

$$T_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } 1 < i < l \text{ and } j = i - 1, \\ 1, & \text{if } i = 1 \text{ and } j = 3, \\ 1, & \text{if } i = l \text{ and } j = l - 2, \\ 0, & \text{otherwise} \end{cases}$$

or

$$T = \begin{pmatrix} 1 & 0 & 1 & & & & & & & \\ 1 & 1 & 0 & & & & & & & \\ 0 & 1 & 1 & & & & & & & \\ & & 1 & 1 & & & & & & \\ & & & \cdot & \cdot & & & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & \cdot & \cdot & & & \\ & & & & & & 1 & 1 & & \\ & & & & & & & 1 & 1 & 0 \\ & & & & & & & 1 & 0 & 1 \end{pmatrix}.$$

Then we can have a theorem.

THEOREM 3.4. *Let T be the characteristic matrix of a hybrid CA H of length $l(\geq 4)$ with an intermediate boundary condition where the rule applied to the first $l-1$ cells of H is* 60 *and the rule applied to the last* 1 *cell of H is* 102 *or where the rule applied to the first* 1 *cell of H is* 60 *and the rule applied to the last $l-1$ cells of H is* 102*. Then:*

(1) *If $l \leq 4+2^t$ for some non-negative integer t, then $(T^{2^t\cdot 3})^2 = T^{2^t\cdot 3}$.*
(2) *If $4+2^{t-1} < l \leq 4+2^t$ for some positive integer t, then $T^{1+m} = T$ for some positive integer m if and only if m is a multiple of $2^t \cdot 3$.*

Proof. The characteristic matrix of a hybrid CA with the rule applied to the first $l-1$ cells of H is 60 and the rule applied to the last 1 cell of H is 102 is symmetric with the characteristic matrix of a hybrid CA with the rule applied to the first 1 cell of H is 60 and the rule applied to the last $l-1$ cells of H is 102. And periodicities of symmetric matrices are identical each other. So we may assume that the rule applied to the first $l-1$ cells of H is 60 and the rule applied to the last 1 cell of H is 102. Let r be a positive integer and I be the identity matrix. Considering T^r as T^rI, an iteration of multiplication of T is just an iteration of applying row operations to I. So we can easily see that $(l-1)$th row and (l)th row of T^r are identical each other except the last 2 entries. And the last 2 columns of T^r are identical with the last 2 columns of I and not changed with such an iteration. Thus it suffices for the results to consider only the first $l-1$ rows of T which may be identified with the characteristic matrix of a uniform CA of length $l-1$ with rule 60 and an intermediate boundary condition. Hence we have the results by Theorem 3.1 and 3.2. $\square$

Finally, we deal with the characteristic matrix T of a hybrid CA H of length $l(\geq 5)$ with an intermediate boundary condition where the rule applied to the first $l-2$ cells of H is 60 and the rule applied to the last 2 cells of H is 102. Such a matrix T is given by

$$T_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } 1 < i < l-1 \text{ and } j = i-1, \\ 1, & \text{if } i = 1 \text{ and } j = 3, \\ 1, & \text{if } i = l-1 \text{ and } j = l, \\ 1, & \text{if } i = l \text{ and } j = l-2, \\ 0, & \text{otherwise} \end{cases}$$

or

$$T = \begin{pmatrix} 1 & 0 & 1 & & & & & & & \\ 1 & 1 & 0 & & & & & & & \\ 0 & 1 & 1 & & & & & & & \\ & & 1 & 1 & & & & & & \\ & & & \cdot & \cdot & & & & & \\ & & & & \cdot & \cdot & & & & \\ & & & & & \cdot & \cdot & & & \\ & & & & & & 1 & 1 & & \\ & & & & & & & 0 & 1 & 1 \\ & & & & & & & 1 & 0 & 1 \end{pmatrix}.$$

Now we can have a theorem.

THEOREM 3.5. *Let T be the characteristic matrix of a hybrid CA H of length $l(\geq 5)$ with an intermediate boundary condition where the rule applied to the first $l-2$ cells of H is* 60 *and the rule applied to the last* 2 *cells of H is* 102 *or where the rule applied to the first* 2 *cells of H is* 60 *and the rule applied to the last $l-2$ cells of H is* 102. *Then:*

(1) *If $l \leq 3+2^t$ for some positive integer t, then $(T^{2^t\cdot 3})^2 = T^{2^t\cdot 3}$.*
(2) *If $3+2^{t-1} < l \leq 3+2^t$ for some positive integer t, then $T^{1+m} = T$ for some positive integer m if and only if m is a multiple of $2^t \cdot 3$.*

Proof. Similarly to the proof of Theorem 3.4, we may assume that the rule applied to the first $l-2$ cells of H is 60 and the rule applied to the last 2 cells of H is 102 and regard an iteration of multiplication of T as an iteration of applying row operations to I. Now let S be the characteristic matrix of a uniform CA of length l configured with rule 60 and an intermediate boundary condition. And let r be a positive integer. Then the first $l-2$ rows of T^r is the same as the first $l-2$ rows of S^r. And the (l)th row of T^r and $(l-1)$th row of S^r are identical each other except the last 2 entries. Besides, the $(l-1)$th row of T^r and (l)th row of S^r are identical each other except the last 2 entries. We can easily see not only that an $(l-1)$th column of T^r is identical with an $(l-1)$th column of I and not changed with such an iteration but also that the periodicity of an (l)th column of T with the iteration is 2. And the periodicity of S with such an iteration is always even by Theorem 3.1 and 3.2. Thus the periodicity of T is the same as the periodicity of S with such an iteration. Hence we have the results by Theorem 3.1 and 3.2 again. $\square$

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